

Indecomposable 1-factorizations of the complete multigraph λK_{2n} for every $\lambda \leq 2n^*$

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Abstract

A 1-factorization of the complete multigraph λK_{2n} is said to be indecomposable if it cannot be represented as the union of 1-factorizations of $\lambda_0 K_{2n}$ and $(\lambda - \lambda_0) K_{2n}$, where $\lambda_0 < \lambda$. It is said to be simple if no 1-factor is repeated. For every $n \geq 9$ and for every $(n-2)/3 \leq \lambda \leq 2n$, we construct an indecomposable 1-factorization of λK_{2n} which is not simple. These 1-factorizations provide simple and indecomposable 1-factorizations of λK_{2s} for every $s \geq 18$ and $2 \leq \lambda \leq 2\lfloor s/2 \rfloor - 1$. We also give a generalization of a result by Colbourn et al. which provides a simple and indecomposable 1-factorization of λK_{2n} , where $2n = p^m + 1$, $\lambda = (p^m - 1)/2$, p prime.

Keywords: complete multigraph, indecomposable 1-factorizations, simple 1-factorizations.

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1 Introduction

We refer to [3] for graph theory notation and terminology which are not introduced explicitly here. We recall that the complete multigraph λK_{2n} has $2n$ vertices and each pair of vertices is joined by exactly λ edges. A 1-factor of λK_{2n} is a spanning subgraph of λK_{2n} consisting of n edges that are pairwise independent. If \mathcal{S} is a set of 1-factors of λK_{2n} , then we will denote by $E(\mathcal{S})$ the multiset containing all the edges of the 1-factors of \mathcal{S} , namely, $E(\mathcal{S}) = \cup_{F \in \mathcal{S}} E(F)$. A 1-factorization \mathcal{F} of λK_{2n} is a partition

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of the edge-set of λK_{2n} into 1-factors. A subfactorization of \mathcal{F} is a subset \mathcal{F}_0 of 1-factors belonging to \mathcal{F} that constitute a 1-factorization of $\lambda_0 K_{2n}$, where $\lambda_0 \leq \lambda$. For every $\lambda \geq 1$, it is possible to find a 1-factorization of λK_{2n} . Lucas' construction provides a 1-factorization for the complete graph K_{2n} , denoted by GK_{2n} (see [9]). By taking λ copies of GK_{2n} , we find a 1-factorization of λK_{2n} . Obviously, it contains repeated 1-factors. Moreover, we can consider $\lambda_0 < \lambda$ copies of each 1-factor so that it is the union of 1-factorizations of $\lambda_0 K_{2n}$ and $(\lambda - \lambda_0) K_{2n}$. A 1-factorization of λK_{2n} that contains no repeated 1-factors is said to be *simple*. A 1-factorization of λK_{2n} that can be represented as the union of 1-factorizations of $\lambda_0 K_{2n}$ and $(\lambda - \lambda_0) K_{2n}$, where $\lambda_0 < \lambda$, is said to be *decomposable*, otherwise it is called *indecomposable*. An indecomposable 1-factorization might be simple or not. In this paper, we consider the problem about the existence of indecomposable 1-factorizations of λK_{2n} . Obviously, $\lambda > 1$. In order that the complete multigraph λK_{2n} admits an indecomposable 1-factorization, the parameter λ cannot be arbitrarily large: we have necessarily $\lambda < 3 \cdot 4 \cdots (2n - 3)$ or $\lambda < [n(2n - 1)]^{n(2n-1)} \binom{2n^3+n^2-n+1}{2n^2-n}$, according to whether the 1-factorization is simple or not (see [2]). Moreover, two non-existence results are known. For every $\lambda > 1$ there is no indecomposable 1-factorization of λK_4 (see [4]). For every $\lambda \geq 3$ there is no indecomposable 1-factorization of λK_6 (see [2]). We recall that in [4] the authors construct simple and indecomposable 1-factorizations of λK_{2n} for $2 \leq \lambda \leq 12$, $\lambda \neq 7, 11$. They also give a simple and indecomposable 1-factorization of λK_{p+1} , where p is an odd prime and $\lambda = (p - 1)/2$. In [1] we can find an indecomposable 1-factorization of $(n - p) K_{2n}$, where p is the smallest prime not dividing n . This 1-factorization is not simple, but it is used to construct a simple and indecomposable 1-factorization of $(n - p) K_{2s}$ for every $s \geq 2n$. This construction improves the results in [4] for $2 \leq \lambda \leq 12$ (see Theorem 2.5 in [1]). Simple and indecomposable 1-factorizations of $(n - d) K_{2n}$, with $d \geq 2$, $n - d \geq 5$ and $\gcd(n, d) = 1$, are constructed in [8]. Other values of λ and n for which the existence of a simple and indecomposable 1-factorization of λK_{2n} is known are the following: $2n = q^2 + 1$, $\lambda = q - 1$, where q is an odd prime power (see [6]); $2n = 2^h + 2$, $\lambda = 2$ (see [7]); $2n = q^2 + 1$, $\lambda = q + 1$, where q is an odd prime power (see [5]); $2n = q^2$, $\lambda = q$, where q is an even prime power (see [5]).

In this paper we prove some theorems about the existence of simple and indecomposable 1-factorizations of λK_{2n} , where most of the parameters λ and n were not previously considered in literature. We show that for every $n \geq 9$ and for every $(n - 2)/3 \leq \lambda \leq 2n$ there exists an indecomposable 1-factorization of λK_{2n} (see Theorem 1). We can also exhibit some examples of indecomposable 1-factorizations of λK_{2n} for $n \in \{7, 8\}$, $(n - 2)/3 \leq \lambda \leq n$ (see Proposition 3); and for $n \in \{5, 6\}$, $(n - 2)/3 \leq \lambda \leq n - 2$ (see Proposition 1 and 2). The 1-factorizations in Theorem 1, Proposition 1, 2 and 3 are

not simple. By an embedding result in [4], we can use them to prove the existence of simple and indecomposable 1-factorizations of λK_{2s} for every $s \geq 18$ and for every $2 \leq \lambda \leq 2\lfloor s/2 \rfloor - 1$ (see Theorem 2). We note that for odd values of s , the parameter λ does not exceed the value $s - 2$. Nevertheless, if $2s = p^m + 1$, where p is a prime, then we can find a simple and indecomposable 1-factorization of $(s - 1)K_{2s}$ (see Theorem 3). By our results we can improve Theorem 2.5 in [1] about the existence of simple and indecomposable 1-factorizations of λK_{2n} for $2 \leq \lambda \leq 12$. We note that in Theorem 2.5 in [1] the existence of a simple and indecomposable 1-factorization of $11K_{2n}$ (respectively, $12K_{2n}$) is known for every $2n \geq 52$ (respectively, $2n \geq 32$). By Theorem 2, a simple and indecomposable 1-factorization of $11K_{2n}$ exists for every $2n \geq 36$. By Theorem 3, there exists a simple and indecomposable 1-factorization of $12K_{26}$. Moreover, Theorem 3 extends Theorem 2 in [4] to each odd prime power.

2 Basic lemmas.

In Section 3 and 4 we will construct indecomposable 1-factorizations of λK_{2n} for suitable values of $\lambda > 1$. These 1-factorizations contain 1-factor-orbits, that is, sets of 1-factors belonging to the same orbit with respect to a group G of permutations on the vertices of the complete multigraph.

If not differently specified, we use the exponential notation for the action of G and its subgroups on vertices, edges and 1-factors. So, if $e = [x, y]$ is an edge of λK_{2n} and $g \in G$ we set $e^g = [x^g, y^g]$. Analogously, if F is a 1-factor we set $F^g = \{e^g : e \in F\}$. Since we shall treat with sets and multisets, we specify that by an edge-orbit e^H , where $H \leq G$, we mean the set $e^H = \{e^h : h \in H\}$ and by a 1-factor-orbit F^H we mean the set $F^H = \{F^h : h \in H\}$. If $h \in H$ leaves F invariant, that is, $F^h = F$, then h is an element of the stabilizer of F in G , which will be denoted by G_F . The cardinality of F^H is $|H|/|H \cap G_F|$. The following result holds.

Lemma 1. *Let F be a 1-factor of λK_{2n} containing exactly μ edges belonging to the same edge-orbit e^H , where H is a subgroup of G having trivial intersection with the stabilizer of F in G and with the stabilizer of e in G . The multiset $\cup_{h \in H} E(F^h)$ contains every edge of e^H exactly μ times.*

Proof. We denote by e_1, \dots, e_μ the edges in $F \cap e^H$. We show that every edge $f \in e^H$ appears $t_f \geq \mu$ times in the multiset $E(F^H) = \cup_{h \in H} E(F^h)$. For every edge $e_i \in \{e_1, \dots, e_\mu\}$ there exists an element $h_i \in H$ such that $e_i^{h_i} = f$, since e_i and f belong to the same edge-orbit e^H . Hence the 1-factor F^{h_i} contains the edge f . The 1-factors $F^{h_1}, F^{h_2}, \dots, F^{h_\mu}$ are pairwise distinct, since H has trivial intersection with G_F . Therefore, every edge $f \in e^H$ appears $t_f \geq \mu$ times in the multiset $E(F^H)$. We prove that $t_f = \mu$. In fact, $t_f > \mu$ implies the existence of $h \in H \setminus \{h_1, \dots, h_\mu\}$ such that

$f \in F^h$ and then $e_i^{h_i} = f = e_i^h$ for some $e_i \in \{e_1, \dots, e_\mu\}$. That yields a contradiction, since e_i , as well as e , has trivial stabilizer in H . \square

To prove the indecomposability of the 1-factorizations in Section 3, we will use the following lemma.

Lemma 2. *Let M be a 1-factor of λK_{2n} . Let \mathcal{F} be a 1-factorization of λK_{2n} containing $0 \leq \lambda - t < \lambda$ copies of M and a subset \mathcal{S} of 1-factors satisfying the following properties:*

- (i) *the multiset $E(\mathcal{S})$ contains every edge of M exactly t times;*
- (ii) *for every $\mathcal{S}' \subset \mathcal{S}$, the multiset $E(\mathcal{S}')$ contains $0 < \mu < n$ distinct edges of M .*

If $\mathcal{F}_0 \subseteq \mathcal{F}$ is a 1-factorization of $\lambda_0 K_{2n}$, where $\lambda_0 \leq \lambda$, then $\mathcal{S} \subseteq \mathcal{F}_0$ or \mathcal{F}_0 contains no 1-factor of \mathcal{S} .

Proof. Assume that \mathcal{F}_0 contains $0 < s < |\mathcal{S}|$ elements of \mathcal{S} , say F_1, \dots, F_s . We denote by M' the set consisting of the edges of M that are contained in the multiset $\cup_{i=1}^s E(F_i)$. By property (ii), the set M' is a non-empty proper subset of M . It is clear from (i) that the 1-factors of \mathcal{F} containing some edges of M are exactly the $\lambda - t$ copies of M together with the 1-factors of \mathcal{S} . Therefore, the 1-factorization \mathcal{F}_0 contains λ_0 copies of M , since the edges in $M \setminus M'$ are not contained in $\cup_{i=1}^s E(F_i)$. Then the multiset $E(\mathcal{F}_0)$ contains at least $\lambda_0 + 1$ copies of each edge in M' , a contradiction. Hence $s = n$ or \mathcal{F}_0 contains no 1-factor of \mathcal{S} . \square

3 Indecomposable 1-factorizations which are not simple.

In what follows, we consider the group G given by the direct product $\mathbb{Z}_n \times \mathbb{Z}_2$ and denote by H the subgroup of G isomorphic to \mathbb{Z}_n . We will identify the vertices of the complete multigraph λK_{2n} with the elements of G , thus obtaining the graph $\lambda K_G = (G, \lambda \binom{G}{2})$, where $\binom{G}{2}$ is the set of all possible 2-subsets of G and $\lambda \binom{G}{2}$ is the multiset consisting of λ copies of $\binom{G}{2}$.

In G we will adopt the additive notation and observe that G is a group of permutations on the vertex-set, that is, each $g \in G$ is identified with the permutation $x \rightarrow x + g$, for every $x \in G$. For the sake of simplicity, we will represent the elements of G in the form a_j , where a and j are integers modulo n and modulo 2, respectively. The edges of λK_G are of type $[a_0, b_1]$ or $[a_j, b_j]$ and we can observe that each edge $[a_0, b_1]$ has trivial stabilizer in H . For every $a \in \mathbb{Z}_n$, we consider the edge-orbit $M_a = [0_0, a_1]^H$. Each edge-orbit M_a is a 1-factor of λK_G . The 1-factors in $\cup_{a \in \mathbb{Z}_n} M_a$ partition the

edges of type $[a_0, b_1]$. We shall represent the vertices and the 1-factors M_a as in Figure 1. Observe that, if M_a contains the edge $[x_0, (x+a)_1]$, then M_{n-a} contains the edge $[(x+a)_0, x_1]$.

The edges of type $[a_j, b_j]$, with $j = 0, 1$, can be partitioned by the 1-factors (or, near 1-factors) of a 1-factorization (or, of a near 1-factorization) of K_n . More specifically, for even values of n we consider the well-known 1-factorization GK_n defined by Lucas [9]. We recall that in GK_n the vertex-set of K_n is $\mathbb{Z}_{n-1} \cup \{\infty\}$ and $GK_n = \{L_i : i \in \mathbb{Z}_{n-1}\}$, where $L_0 = \{[a, -a] : a \in \mathbb{Z}_{n-1} - \{0\}\} \cup \{[0, \infty]\}$ and $L_i = L_0 + i = \{[a+i, -a+i] : a \in \mathbb{Z}_{n-1} - \{0\}\} \cup \{[i, \infty]\}$. For odd values of n , we consider the 1-factorization GK_{n+1} and delete the vertex ∞ . Each 1-factor L_i yields a near 1-factor L_i^* of K_n where the vertex $i \in \mathbb{Z}_n$ is unmatched. We denote by GK_n^* the resulting near 1-factorization of K_n .

For even values of n , we partition the edges $[a_j, b_j]$ of λK_{2n} into 1-factors of λK_{2n} as follows. For $j = 0, 1$, we consider the 1-factorization GK_n of the complete graph K_n with vertex-set $V_j = \{a_j : 0 \leq a \leq n-1\}$. It is possible to obtain a 1-factor of K_{2n} by joining, in an arbitrary way, a 1-factor on V_0 to a 1-factor on V_1 . We denote by $\mathcal{F}(GK_n)$ the resulting set of 1-factors of K_{2n} . We denote by $\mathcal{F}(\lambda GK_n)$ the multiset consisting of λ copies of $\mathcal{F}(GK_n)$.

For odd values of n , we partition the edges $[a_j, b_j]$ of λK_{2n} into 1-factors of λK_{2n} as follows. For $j = 0, 1$, we consider the near 1-factorization GK_n^* of the complete graph K_n with vertex-set $V_j = \{a_j : 0 \leq a \leq n-1\}$. We select an integer $b \in \mathbb{Z}_n$. For $i = 0, \dots, n-1$, we join the near 1-factor L_i^* on V_0 to the near 1-factor L_{i+b}^* on V_1 (subscripts are considered modulo n) and add the edge $[i_0, (i+b)_1]$. We obtain a 1-factor of K_{2n} . We denote by $\mathcal{F}(GK_n^*, b)$ the resulting set of 1-factors of K_{2n} . We denote by $\mathcal{F}(\lambda GK_n^*, b)$ the multiset consisting of λ copies of $\mathcal{F}(GK_n^*, b)$. Observe that the set $\{[i_0, (i+b)_1] : 0 \leq i \leq n-1\}$ corresponds to the 1-factor M_b . Hence $\mathcal{F}(\lambda GK_n^*, b)$ contains every edge of M_b exactly λ times.

In the following propositions we will construct 1-factorizations of λK_G which are not simple. They are obtained as described in Lemma 3. Moreover, Lemma 4 will be useful to prove that these 1-factorizations are indecomposable. It is straightforward to prove that the following holds.

Lemma 3. *Let $\mathcal{F}' = \{F_1, \dots, F_m\}$ be a set of 1-factors of λK_G such that each F_i contains no edge of type $[a_j, b_j]$, has trivial stabilizer in H and $F_r \notin F_i^H$ for each pair (i, r) with $i \neq r$.*

Let \mathcal{M} be the subset of $\{M_a : a \in \mathbb{Z}_n\}$ containing all the 1-factors M_a such that $t(M_a) = \sum_{i=1}^m |E(M_a) \cap E(F_i)| > 0$.

If $|H| = n$ is even and $t(M_a) \leq \lambda$ for every $M_a \in \mathcal{M}$, then there exists a 1-factorization of λK_G whose 1-factors are exactly those of $F_1^H \cup \dots \cup F_m^H \cup \mathcal{F}(\lambda GK_n)$ together with $\lambda - t(M_a)$ copies of each $M_a \in \mathcal{M}$ and λ copies of each $M_a \notin \mathcal{M}$.

If $|H| = n$ is odd, $t(M_a) \leq \lambda$ for every $M_a \in \mathcal{M}$ and there exists at least

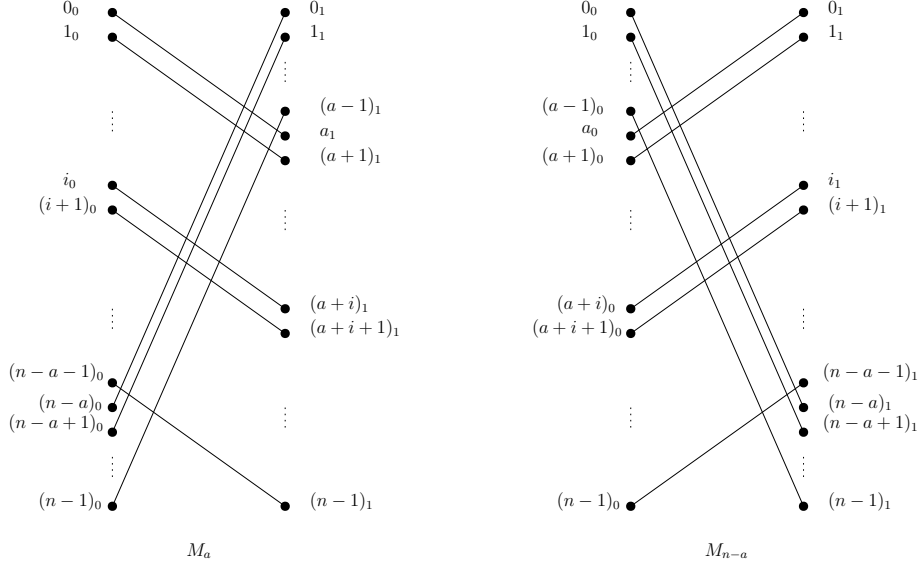


Figure 1: The vertices a_0, b_1 of λK_G are represented on the left and on the right, respectively. Each edge-orbit M_a is a 1-factor of λK_G . If M_a contains the edge $[0_0, a_1]$, then M_{n-a} contains the edge $[a_0, 0_1]$.

one 1-factor $M_b \in \{M_a : a \in \mathbb{Z}_n\} \setminus \mathcal{M}$, then there exists a 1-factorization of λK_G whose 1-factors are exactly those of $F_1^H \cup \dots \cup F_m^H \cup \mathcal{F}(\lambda GK_n, b)$ together with $\lambda - t(M_a)$ copies of each $M_a \in \mathcal{M}$ and λ copies of each $M_a \notin \mathcal{M} \cup \{M_b\}$. \square

Lemma 4. Let \mathcal{F} be the 1-factorization of λK_G obtained in Lemma 3 starting from $\mathcal{F}' = \{F_1, \dots, F_m\}$ and the set \mathcal{M} . Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a 1-factorization of $\lambda_0 K_G$, $\lambda_0 \leq \lambda$. Let $F_i \in \mathcal{F}'$ and $M_a \in \mathcal{M}$ be such that F_i contains exactly one edge of M_a . If one of the following conditions holds:

- (i) each 1-factor in $\mathcal{F}' \setminus \{F_i\}$ contains no edge of M_a ;
- (ii) each 1-factor $F \in \mathcal{F}' \setminus \{F_i\}$ containing some edge of M_a is such that either $F^H \subset \mathcal{F}_0$ or $F^H \cap \mathcal{F}_0 = \emptyset$.

then it is either $F_i^H \subset \mathcal{F}_0$ or $F_i^H \cap \mathcal{F}_0 = \emptyset$.

Proof. Assume that F_i satisfies property (i). By Lemma 1, each edge of M_a appears exactly once in the multiset $E(F_i^H)$. Since each 1-factor in $\mathcal{F}' \setminus \{F_i\}$ contains no edge of M_a , the 1-factorization \mathcal{F} contains exactly $\lambda - 1$ copies of M_a . The assertion follows from Lemma 2 by setting $\mathcal{S} = F_i^H$ and $M = M_a$.

Assume that F_i satisfies property (ii). We can consider the subset \mathcal{F}_1 of $\mathcal{F}' \setminus \{F_i\}$ consisting of the 1-factors F containing $s_F \geq 1$ edges of M_a and whose orbit F^H is contained in \mathcal{F}_0 . The set \mathcal{F}_1 might be empty. By Lemma

1, each edge of M_a appears exactly $s_F \geq 1$ times in the multiset $E(F^H)$, where $F \in \mathcal{F}_1$. Hence $\lambda_0 \geq \sum_{F \in \mathcal{F}_1} s_F \geq 0$ (if $\mathcal{F}_1 = \emptyset$, then $\sum_{F \in \mathcal{F}_1} s_F = 0$). Set $\mathcal{S} = F_i^H \cap \mathcal{F}_0$ and suppose that $0 < |\mathcal{S}| < n$, where $n = |F_i^H|$. Let M' be the subset of M_a consisting of the edges of M_a that are contained in the multiset $E(\mathcal{S})$. By the proof of Lemma 1, the set M' consists of $|\mathcal{S}| < n$ distinct edges. Each edge of $M_a \setminus M'$ appears exactly $\sum_{F \in \mathcal{F}_1} s_F \leq \lambda_0$ times among the edges of the 1-factors in $\mathcal{S} \cup (\cup_{F \in \mathcal{F}_1} F^H)$. Each edge of M' appears exactly $1 + \sum_{F \in \mathcal{F}_1} s_F$ times among the edges of the 1-factors in $\mathcal{S} \cup (\cup_{F \in \mathcal{F}_1} F^H)$. Whence $\sum_{F \in \mathcal{F}_1} s_F < \lambda_0$, otherwise the edges of M' would appear at least $\lambda_0 + 1$ times among the edges of the 1-factors in \mathcal{F}_0 . Since the edges of $M_a \setminus M'$ appear $\sum_{F \in \mathcal{F}_1} s_F < \lambda_0$ times, the 1-factorization \mathcal{F}_0 must contain $\lambda_0 - \sum_{F \in \mathcal{F}_1} s_F > 0$ copies of M_a . Consequently, each edge of M' appears at least $\lambda_0 + 1$ among the edges of the 1-factors in \mathcal{F}_0 . That yields a contradiction. Hence, either \mathcal{F}_0 contains no 1-factor of F_i^H or $\mathcal{F}_i^H \subseteq \mathcal{F}_0$. \square

Proposition 1. *Let $n \geq 5$ and $(n - 2)/3 \leq \lambda \leq n - 2$ such that $n - \lambda$ is even. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. Identify λK_{2n} with λK_G . If $\lambda < n - 2$, then $n > 5$ and we consider the 1-factor A in Figure 2(a). For $\lambda = n - 2$ we consider the 1-factor A in Figure 3 with $\alpha = 1$. If $\lambda < n - 2$, then A contains exactly $(n - \lambda - 2)/2$ edges of M_1 as well as $(n - \lambda - 2)/2$ edges of M_{n-1} . It also contains λ edges of M_0 , one edge of M_2 and one edge of M_{n-2} . If $\lambda = n - 2$, then A contains exactly λ edges of M_0 as well as one edge of M_1 and one edge of M_{n-1} . In both cases the stabilizer of A in H is trivial and when $\lambda < n - 2$, the condition $(n - 2)/3 \leq \lambda$ assures that $(n - \lambda - 2)/2 \leq \lambda$. Therefore $\mathcal{F}' = \{A\}$ satisfies Lemma 3 and a 1-factorization \mathcal{F} of λK_G is constructed as prescribed. We prove that \mathcal{F} is indecomposable. Suppose that $\mathcal{F}_0 \subseteq \mathcal{F}$ is a 1-factorization of $\lambda_0 K_G$, $\lambda_0 < \lambda$. The 1-factor A satisfies condition (i) of Lemma 4 (set $M_a = M_2$ or $M_a = M_1$ according to whether $\lambda < n - 2$ or $\lambda = n - 2$, respectively). Therefore it is either $A^H \subset \mathcal{F}_0$ or $A^H \cap \mathcal{F}_0 = \emptyset$. In the former case, each edge of M_0 appears λ times in the multiset $E(\mathcal{F}_0)$, that is, $\lambda = \lambda_0$, a contradiction. In the latter case, no edge of M_0 appears in $E(\mathcal{F}_0)$, a contradiction. \square

Proposition 2. *Let $n \geq 5$ and $(n + 1)/3 \leq \lambda \leq n - 3$ such that $n - \lambda$ is odd. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. The proof is similar to the proof of Proposition 1. \square

Proposition 3. *Let $n \geq 7$ and $n - 1 \leq \lambda \leq n$. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

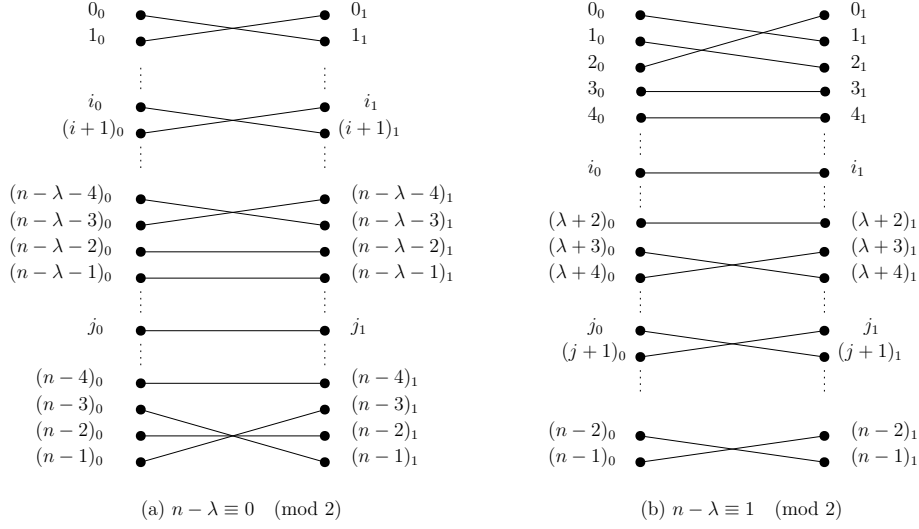


Figure 2: The 1-factor A in the case: (a) $n - \lambda$ even, $\lambda < n - 2$; (b) $n - \lambda$ odd

Proof. Identify λK_{2n} with λK_G and set $\lambda = n - 1 + r$, where $0 \leq r \leq 1$. We consider the 1-factors A and B_r in Figure 3. In the definition of A , we set $\alpha = 3$ if $r = 0$; $\alpha = 2$ if $r = 1$. The 1-factors A, B_r have trivial stabilizer in H . Moreover, the multiset $E(A) \cup E(B_r)$ is contained in the multiset $E(\mathcal{M})$, where $\mathcal{M} = \{M_0, M_1, M_\alpha, M_{n-\alpha}, M_{r+2}\}$. We note that the 1-factors in \mathcal{M} are pairwise distinct, since $n \geq 7$. Whence $t(M_a) = |E(M_a) \cap A| + |E(M_a) \cap E(B_r)| \leq \lambda$ for every $M_a \in \mathcal{M}$. More specifically, $t(M_0) = (n - 2) + (r + 1) = \lambda$, $t(M_1) = n - r - 2 = \lambda - 1$, $t(M_a) = 1$ for every $a \in \{\alpha, n - \alpha, r + 2\}$. By Lemma 3, we construct a 1-factorization \mathcal{F} of λK_G that contains $A^H \cup B_r^H$.

We prove that \mathcal{F} is indecomposable. Firstly, note that if $\mathcal{F}_0 \subseteq \mathcal{F}$ is a 1-factorization of $\lambda_0 K_G$, $\lambda_0 < \lambda$, then $F^H \subset \mathcal{F}_0$ or $F^H \cap \mathcal{F}_0 = \emptyset$ for $F \in \{A, B_r\}$. This follows from Lemma 4 by observing that A and M_α satisfy condition (i). The same can be repeated for B_r and M_{r+2} . If $A^H \subset \mathcal{F}_0$ and $B_r^H \subset \mathcal{F}_0$, then each edge of M_0 appears λ times in the multiset $E(\mathcal{F}_0)$ and then $\lambda_0 = \lambda$, a contradiction. In the same manner, if $A^H \cap \mathcal{F}_0 = B_r^H \cap \mathcal{F}_0 = \emptyset$, then no edge of M_0 appears in the multiset $E(\mathcal{F}_0)$, a contradiction. Therefore, exactly one of the orbits A^H, B_r^H is contained in \mathcal{F}_0 . Without loss of generality, we can assume that $A^H \subset \mathcal{F}_0$ and $B_r^H \cap \mathcal{F}_0 = \emptyset$. Each edge of M_0 appears at least $n - 2$ times in the multiset $E(\mathcal{F}_0)$, that is, $\lambda_0 \geq n - 2$. Each edge of M_1 appears at least $n - 2 - r$ in the multiset $E(\mathcal{F} \setminus \mathcal{F}_0)$, that is, $\lambda - \lambda_0 \geq n - 2 - r$. By summing up these two relations, we have $\lambda \geq 2n - 4 - r$ and since $\lambda \leq n$, this yields $n \leq 5$, a contradiction. \square

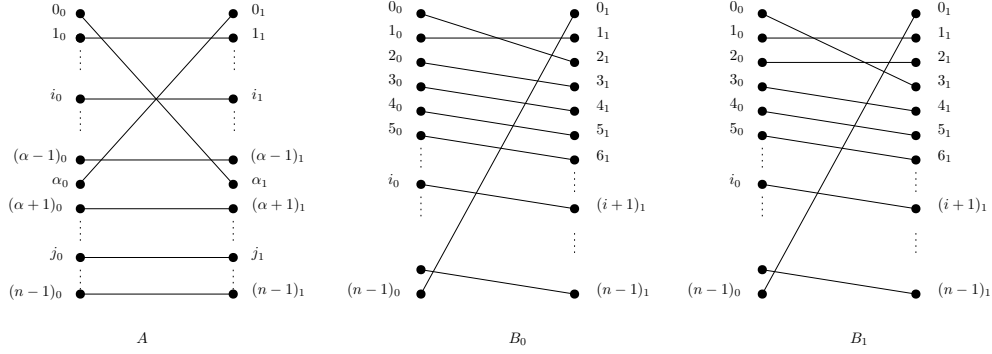


Figure 3: The 1-factors A and B_r , $r = 0, 1$, defined in the proof of Proposition 3.

Proposition 4. *Let $n \geq 9$ and $n + 1 \leq \lambda \leq 2n - 8$. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. Identify λK_{2n} with λK_G . We distinguish the cases $n \neq 11$ and $n = 11$. For $n \neq 11$, we set $\lambda = n + r$, where $1 \leq r \leq n - 8$, and consider the 1-factors A and $B = B_0$ in Figure 3. In the definition of A we set $\alpha = 3$. We also define the 1-factors C and D in Figure 4.

For $n = 11$, we set $\lambda = 9 + r$, where $3 \leq r \leq 5$. We consider the 1-factor A in Figure 3, where $\alpha = 2$ or $\alpha = 3$, according to whether $r = 3, 4$ or $r = 5$, respectively. For $r = 3, 4$ we also consider the 1-factor $B = \{[i_0, i_1] : 1 \leq i \leq r\} \cup \{[i_0, (i+1)_1] : r+1 \leq i \leq 10\} \cup \{[0_0, (r+1)_1]\}$. For $r = 5$, we consider the 1-factor $B = B_0$ in Figure 3 and the 1-factor $C = \{[i_0, i_1] : 1 \leq i \leq 4\} \cup \{[i_0, (i+1)_1] : 5 \leq i \leq 10, i \neq 6\} \cup \{[0_0, 7_1], [6_0, 5_1]\}$. We can construct a 1-factorization \mathcal{F} of λK_G as described in Lemma 3. By Lemma 4, the 1-factorization \mathcal{F} is indecomposable. The proof is similar to that of Proposition 3 \square

Proposition 5. *Let $n \geq 9$ and $\lambda = 2n - 7$. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. We set $\lambda = n + r$ with $r = n - 7$ and consider the 1-factors in $\mathcal{F}' = \{A, B, C, D\}$, where A and $B = B_0$ are described in Figure 3. In the definition of A we set $\alpha = 3$. The 1-factors C and D are defined in Figure 4. The assertion follows from Lemma 4. \square

Proposition 6. *Let $n \geq 9$ and $2n - 6 \leq \lambda \leq 2n - 3$. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. Identify λK_{2n} with λK_G and set $\lambda = 2n - r$, where $3 \leq r \leq 6$. We consider the 1-factors A and $B = B_1$ in Figure 3. In the definition of the

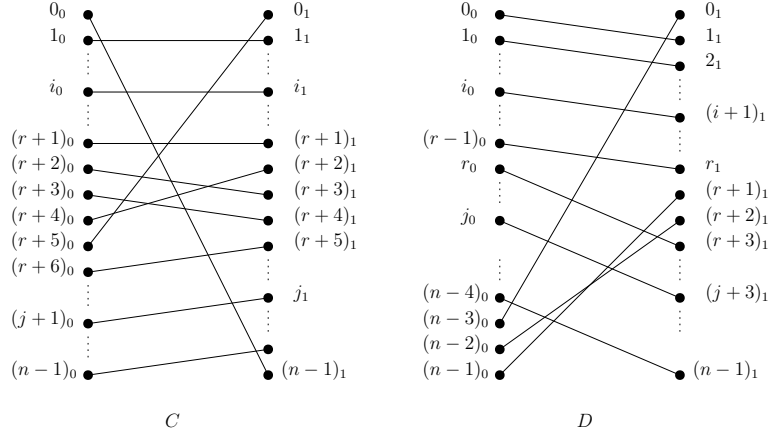


Figure 4: The 1-factors C and D defined in the proof of Proposition 4.

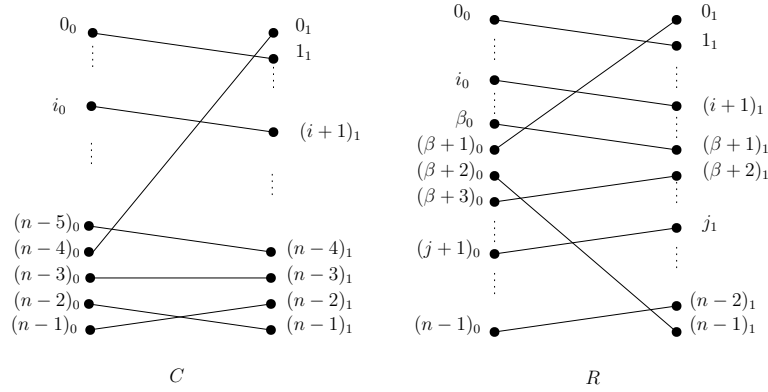


Figure 5: The 1-factors C and R defined in the proof of Proposition 6 and 8, respectively.

1-factor A , the parameter α assumes the value $\alpha = 2$ if $r \in \{3, 5, 6\}$; $\alpha = 4$ if $r = 4$. We define the 1-factor C as in Figure 5. We also consider the 1-factor D_r in Figure 6 for $r = 3, 4$ and in Figure 7 for $r = 5, 6$. We can apply Lemma 3 and construct a 1-factorization \mathcal{F} of λK_G as prescribed. By Lemma 4, we can prove that \mathcal{F} is indecomposable. \square

Proposition 7. *Let $n \geq 9$ and $\lambda = 2n - 2$. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. Identify λK_{2n} with λK_G . We distinguish the cases $n \geq 11$ and $n = 9, 10$. For $n \geq 11$ we consider the 1-factor A in Figure 3 with $\alpha = 2$ and the 1-factor $B_1 = D$. We also consider the 1-factors B, C in Figure 8.

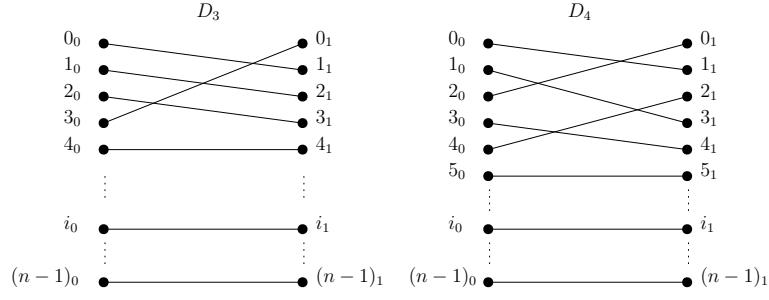


Figure 6: The 1-factor D_r , $r = 3, 4$, defined in the proof of Proposition 6.

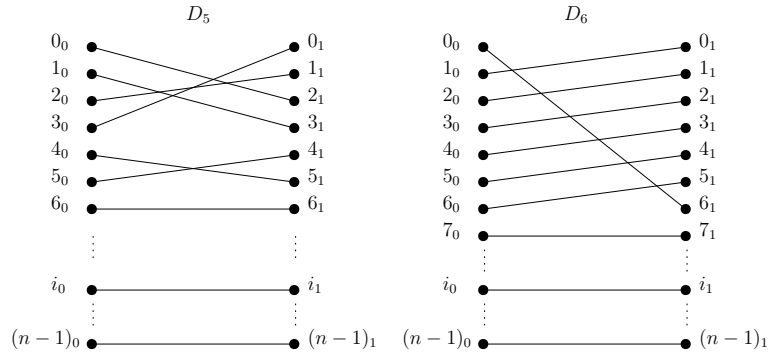


Figure 7: The 1-factor D_r , $r = 5, 6$, defined in the proof of Proposition 6.

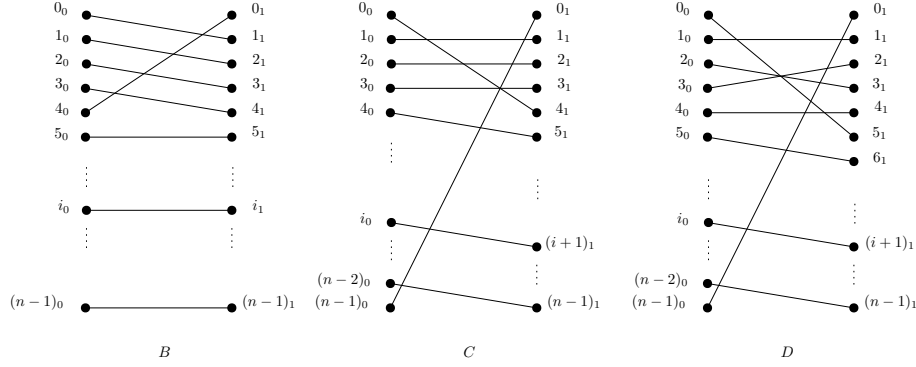


Figure 8: The 1-factors B , C defined in the proof of Proposition 7 and the 1-factor D defined in the proof of Proposition 8 for $\lambda = 2n$

For $n = 9, 10$, we consider two copies of the 1-factor A in Figure 3. We denote by A the copy with $\alpha = 2$ and by B the copy with $\alpha = 3$ or 4 , according to whether $n = 10$ or $n = 9$, respectively. We consider the 1-factors C , D and R_n , where $C = \{[i_0, (i+1)_1] : 2 \leq i \leq n-1\} \cup \{[0_0, 2_1], [1_0, 1_1]\}$; $D = \{[i_0, (i+1)_1] : 2 \leq i \leq n-3\} \cup \{[0_0, (n-1)_1], [(n-1)_0, 0_1], [(n-2)_0, 2_1], [1_0, 1_1]\}$. $R_9 = \{[i_0, (i+1)_1] : 0 \leq i \leq 2\} \cup \{[i_0, (i+2)_1] : 3 \leq i \leq 7\} \cup \{[8_0, 4_1]\}$; $R_{10} = \{[i_0, (i+1)_1] : 0 \leq i \leq 2\} \cup \{[i_0, (i+2)_1] : 3 \leq i \leq 7\} \cup \{[8_0, 0_1], [9_0, 4_1]\}$. We can construct a 1-factorization \mathcal{F} as described in Lemma 3. By Lemma 4, we can prove that \mathcal{F} is indecomposable. \square

Proposition 8. *Let $n \geq 9$ and $2n - 1 \leq \lambda \leq 2n$. There exists an indecomposable 1-factorization of λK_{2n} which is not simple.*

Proof. Identify λK_{2n} with λK_G . We consider two copies of the 1-factor A in Figure 3. We denote by A the copy with $\alpha = 2$ ($\alpha = 4$ if $n = 9$ and $\lambda = 18$) and by B the copy with $\alpha = 3$. We also consider the 1-factors C , D , R . For $n \geq 9$ and $(n, \lambda) \neq (9, 18)$, the 1-factor C corresponds to the 1-factor B_1 in Figure 3. For $(n, \lambda) = (9, 18)$ it corresponds to the 1-factor C in Figure 8. For $n \geq 9$ and $\lambda = 2n - 1$, the 1-factor D corresponds to the 1-factor B_0 in Figure 3. For $n > 9$ and $\lambda = 2n$, the 1-factor D is defined in Figure 8. For $n = 9$ and $\lambda = 2n$, it corresponds to the 1-factor B_0 in Figure 3. For $n \geq 9$ and $(n, \lambda) \neq (9, 18)$, the 1-factor R is defined in Figure 5. In the definition of R we set $\beta = 3$ or $\beta = 4$ according to whether $\lambda = 2n - 1$ or $\lambda = 2n$, respectively ($\beta = 5$ if $n = 10$ and $\lambda = 2n$). For $(n, \lambda) = (9, 18)$, we set $R = \{[i_0, (i+1)_1] : 0 \leq i \leq 4, i = 8\} \cup \{[i_0, (i+2)_1] : 5 \leq i \leq 6\} \cup \{[7_0, 6_1]\}$. We construct a 1-factorization \mathcal{F} of λK_G as described in Lemma 3. By Lemma 4, we can prove that \mathcal{F} is indecomposable. \square

Combining the constructions in the previous propositions, the following result holds.

Theorem 1. *Let $n \geq 9$. For every $(n-2)/3 \leq \lambda \leq 2n$ there exists an indecomposable 1-factorization of λK_{2n} which is not simple.* \square

4 Simple and indecomposable 1-factorizations.

In this section we use Theorem 1 and Corollary 4.1 in [4] to find simple and indecomposable 1-factorizations of λK_{2n} . We also generalize the result in [4] about the existence of simple and indecomposable 1-factorizations of λK_{2n} , where $2n-1$ is a prime and $\lambda = (n-1)/2$. We recall the statement of Corollary 4.1 .

Corollary 4.1. *[4] If there exists an indecomposable 1-factorization of λK_{2n} with $\lambda \leq 2n-1$, then there exists a simple and indecomposable 1-factorization of λK_{2s} for $s \geq 2n$.*

The following results hold.

Theorem 2. *Let $s \geq 18$. For every $2 \leq \lambda \leq 2\lfloor s/2 \rfloor - 1$ there exists a simple and indecomposable 1-factorization of λK_{2s} .*

Proof. For every $n \geq 9$ we set $I_n = \{\lambda \in \mathbb{Z} : (n-2)/3 \leq \lambda \leq 2n-1\}$ and note that $I_n \cup I_{n+1} = \{\lambda \in \mathbb{Z} : (n-2)/3 \leq \lambda \leq 2(n+1)-1\}$. Consider $s \geq 2n \geq 2 \cdot 9$. By Corollary 4.1 of [4], for every $\lambda \in I_n$ there exists a simple and indecomposable 1-factorization of λK_{2s} . Since we can consider $9 \leq n \leq \lfloor s/2 \rfloor$, we obtain a simple and indecomposable 1-factorization of λK_{2s} for every $\lambda \in \bigcup_{n=9}^{\lfloor s/2 \rfloor} I_n = \{\lambda \in \mathbb{Z} : 7/3 \leq \lambda \leq 2\lfloor s/2 \rfloor - 1\}$. Since $s \geq 2 \cdot 5$, from Proposition 2 and Corollary 4.1 we also obtain a simple and indecomposable 1-factorization of λK_{2s} for $\lambda = 2$. Hence the assertion follows. \square

Theorem 3. *Let $2n-1$ be a prime power and let $\lambda = n-1$. There exists a simple and indecomposable 1-factorization of λK_{2n} .*

Proof. Let $2n-1 = p^m$, with p an odd prime and $m \geq 1$. Let $GF(p^m)$ be the Galois field of order p^m and let v be a generator of the cyclic multiplicative group $GF(p^m)^* = GF(p^m) - \{0\}$. It is well known that v is a root of an irreducible polynomial over \mathbb{Z}_p of degree m , the field $GF(p^m)$ is an algebraic extension of \mathbb{Z}_p and it is $GF(p^m) = \mathbb{Z}_p(v) = \{a_0 + a_1v + a_2v^2 + \cdots + a_{m-1}v^{m-1} \mid a_i \in \mathbb{Z}_p\}$. Let $V = GF(p^m) \cup \{\infty\}$, $\infty \notin GF(p^m)$, and identify the vertices of the complete multigraph $(n-1)K_{2n}$ with the elements of V , thus the edges are in the multiset $(n-1)\binom{V}{2}$. The affine linear group $AGL(1, p^m) = \{\phi_{b,a} : a, b \in GF(p^m), b \neq 0\}$ is a permutation group on V where each $\phi_{b,a}$ fixes ∞ and maps $x \in V \setminus \{\infty\}$ onto $xb + a$. This action extends to edges and 1-factors. For each edge $e = [x, y]$ and for each 1-factor F , we set $e^{\phi_{b,a}} = eb + a = [xb + a, yb + a]$ and $F^{\phi_{b,a}} = Fb + a$.

If $x \neq \infty$ and $y \neq \infty$ we call $\partial e = \{\pm(y-x)\}$ the *difference set* of e .

Consider the following set of edges:

$$A_0 = \{[(2i-1) + a_1v + a_2v^2 + \cdots + a_{m-1}v^{m-1}, 2i + a_1v + \cdots + a_{m-1}v^{m-1}], \\ 1 \leq i \leq (p-1)/2, a_r \in \mathbb{Z}_p, 1 \leq r \leq m-1\}$$

$$A_1 = \{[(2i-1)v + a_2v^2 + \cdots + a_{m-1}v^{m-1}, (2i)v + a_2v^2 + \cdots + a_{m-1}v^{m-1}], \\ 1 \leq i \leq (p-1)/2, a_r \in \mathbb{Z}_p, 2 \leq r \leq m-1\}$$

$$A_2 = \{[(2i-1)v^2 + \cdots + a_{m-1}v^{m-1}, (2i)v^2 + \cdots + a_{m-1}v^{m-1}], \\ 1 \leq i \leq (p-1)/2, a_r \in \mathbb{Z}_p, 3 \leq r \leq m-1\}$$

...

$$A_{m-1} = \{[(2i-1)v^{m-1}, (2i)v^{m-1}], 1 \leq i \leq (p-1)/2\}.$$

Obviously if $m = 1$ we just have $\mathbb{Z}_p(v) = \mathbb{Z}_p$ and we just take the set A_0 .

Observe that each set A_j , $j = 0, \dots, m-1$, contains exactly $p^{m-j-1}(p-1)/2$ edges with difference set $\{\pm v^j\}$. Let F be the 1-factor given by: $\{[0, \infty]\} \cup A_0 \cup A_1 \cup \dots \cup A_{m-1}$. The set $\mathcal{F} = F^{AGL(1, p^m)}$ is a simple and indecomposable 1-factorization of $(n-1)K_{2n}$. \square

5 Conclusions.

Our methods of construction can be used to obtain indecomposable 1-factorizations of λK_{2n} for some values of $\lambda > 2n$. These 1-factorizations are not simple and do not provide simple 1-factorizations, since for these values of λ we cannot apply Corollary 4.1 of [4].

As remarked in Section 1, a necessary condition for the existence of an indecomposable 1-factorization of λK_{2n} is $\lambda < [n(2n-1)]^{n(2n-1)} \binom{2n^3+n^2-n+1}{2n^2-n}$. It would be interesting to know whether for every $n \geq 4$ there exists a parameter $\lambda(n) < [n(2n-1)]^{n(2n-1)} \binom{2n^3+n^2-n+1}{2n^2-n}$ depending from n such that for every $\lambda > \lambda(n)$ there is no indecomposable 1-factorization of λK_{2n} .

References

- [1] D. Archdeacon, and J.H. Dinitz, Constructing indecomposable 1-factorizations of the complete multigraph, Discrete Math 92 (1991), 9–19.
- [2] A.H. Baartmans, and W.D. Wallis, Indecomposable factorizations of multigraphs, Discrete Math 78 (1989), 37–43.
- [3] J.A. Bondy, and U.S.R. Murty, Graph Theory, Springer-Verlag, London, 2008.

- [4] C.J. Colbourn, M.J. Colbourn, and A. Rosa, Indecomposable 1-factorizations of the complete multigraph, *J Austral Math Soc Ser A* 39 (1985), 334–343.
- [5] Gy. Kiss, One-factorizations of complete multigraphs and quadrics in $PG(n, q)$, *J Combin Des* 10 (2002), 139–143.
- [6] G. Korchmáros, A. Siciliano, and A. Sonnino, 1-factorizations of complete multigraphs arising from finite geometry, *J Combin Theory Ser A* 93 (2001), 385–390.
- [7] A. Sonnino, One-factorizations of complete multigraphs arising from maximal arcs in $PG(2, 2^h)$, *Discrete Math* 231 (2001), 447–451.
- [8] C. Wensong, New constructions of simple and indecomposable 1-factorizations of complete multigraphs, *J Stat Plann Inference* 94 (2001), 181–196.
- [9] E. Lucas, *Récréations mathématiques*, 2, Gauthier-Villars, Paris, 1883, 161–197.